Section 1: n=4

Recall Theorem 1.2.5. For any primitive Pythagorean triple (x, y, z), if we assume x is odd, then there exist relatively prime odd integers s, t such that

$$x = st, \quad y = \frac{s^2 - t^2}{2}, \quad z = \frac{s^2 + t^2}{2}.$$

There is a similar (but different) set of formulae if we assume x is even.

Exercise 2.1.1 Let (x, y, z) be a primitive Pythagorean triple. Assuming x is even, show that there exist relatively prime integers (not necessarily odd) u, v such that

$$x = 2uv, \quad y = u^2 - v^2, \quad z = u^2 + v^2.$$

Pierre Fermat was a remarkable mathematician. As we have mentioned, he was well known not for proving and publishing many results, but for challenging his friends and colleagues to prove a result he claimed to have already proved. In fact, in all the mathematical work left by Fermat, there is only one proof completely written out. It naturally uses his method of infinite descent.

Theorem 2.1.2 The area of an integral right triangle cannot be a square.

Proof: (Essentially this is Fermat's proof) Suppose to the contrary that $x^2 + y^2 = z^2$ and $\frac{1}{2}(xy) = m^2$ for some m. Using Exercise 2.1.1, we know there exist relatively prime integers u and v such that

$$x = 2uv, \quad y = u^2 - v^2, \quad z = u^2 + v^2.$$

Note that one of u and v must be even and one must be odd. (WHY?) So the area of this triangle is given by

$$\begin{aligned} m^2 &= \tfrac{1}{2}(xy) \\ &= \tfrac{1}{2}(2uv)(u^2 - v^2). \\ &= uv(u-v)(u+v) \end{aligned}$$

These four factors are relatively prime (WHY?) and their product is equal to a square, so they must all be squares to begin with. Namely,

$$u = a^2, \ v = b^2, \ u - v = c^2, \ u + v = d^2$$

for some pair-wise relatively prime integers a,b,c,d. Note that c and d are odd and relatively prime. (WHY?) We have $d^2 = c^2 + 2b^2$ (WHY?) which means that

$$2b^2 = d^2 - c^2 = (d-c)(d+c).$$

A few things need to be noted:

(a) $d-c$ and $d+c$ are both even
(b) The greatest common divisor of $d-c$ and $d+c$ is 2
(c) The product $(d-c)(d+c)$ has at least two factors of 2 in it.
(d) The left-hand side $2b^2$ has an odd number of 2's in its factorization.

Therefore,
(e) One of $d-c, d+c$ must have at least two 2's in its factorization and the other factor has exactly one 2.

So we see that there exists integers r and s such that (without loss of generality) $d-c = 4r^2$ and $d+c = 2s^2$. Solving for c and d, we get

$$c = s^2 - 2r^2 \qquad d = s^2 + 2r^2,$$

and hence $b = 2rs$ (WHY?). Consequently, $a^2 = u = \frac{1}{2}(c^2 + d^2) = s^4 + 4r^4$ and we see that $(s^2, 2r^2, a)$ is another primitive Pythagorean triple whose area ($\frac{1}{2}(s^2 2r^2) = (rs)^2$) is a square. To use Fermat's method of infinite descent, we only need to show that a is strictly smaller than z. (Left as exercise.)

Exercise 2.1.3 There are five "WHY?'s" in the previous proof. Provide each reason.

Exercise 2.1.4 In the previous proof, verify that $a < z$.

So Fermat showed that the area of an integral right triangle cannot be a square. What does that have to do with his famous theorem and the case when $n = 4$? It can be shown to be equivalent…as follows:

Theorem 2.1.5 The equation $x^4 + y^4 = z^4$ has no nontrivial integral solutions.

Proof: Suppose (x, y, z) forms a primitive solution (i.e. pair-wise relatively prime). As with the Pythagorean Theorem, one of x or y must be even. Without loss of generality, suppose it is x. From $x^4 = z^4 - y^4 = (z^2 - y^2)(z^2 + y^2)$ and the fact that x is even, we can deduce that there exist relatively prime integers a and b for which

$$z^2 - y^2 = 8a^4 \text{ and } z^2 + y^2 = 2b^4.$$

Solving for z^2, we have $z^2 = 4a^4 + b^4$. But that implies that $(2a^2, b^2, z)$ forms a Pythagorean triangle whose area is a square. That contradicts Theorem 2.1.2, so our assumption is false and there are no solutions to $x^4 + y^4 = z^4$.

It can also be shown (again using Fermat's method of infinite descent) that the equation $x^4 + y^4 = z^2$ has no nontrivial integral solutions. In the next section we will show $x^3 + y^3 = z^3$ also has no nontrivial integral solutions.

Exercise 2.1.6 Consider the equation $x^3 + y^3 = z^2$. Orienting the solutions so that $1 \le x \le y \le z$, one solution is the triple $(2, 2, 4)$. Find two more.